Sagawa and Ueda Reply: In the preceding Comment [1], Dillenschneider and Lutz claim that “the initial state considered in [2] is not an equilibrium state—it’s entropy and free energy are therefore not defined,” and that “comparing the nonequilibrium result of Ref. [2] with the equilibrium result of Landauer is therefore inconsistent.” However, Landauer’s principle can be discussed for nonequilibrium as well as equilibrium states in terms of the Shannon entropy (SE) for classical cases and the von Neumann entropy (VNE) for quantum cases [3,4]. In fact, it has been established [5–8] that, even when the initial and final states of the system are out of equilibrium, \[ \Delta S \equiv \beta Q \tag{1} \]

holds, where \( Q \) is the heat absorbed by a thermodynamic system that is in contact with a single heat bath, and \( \Delta S \) is the change in SE or VNE. Our main result (inequality (3) in [2]) can be derived also from (1), as shown below. For the sake of completeness, we also give a proof of (1) at the end.

We note that in the case of a quantum system that is strongly interacting with a heat bath, we need to renormalize thermodynamic quantities to formulate inequality (1) [9].

In Fig. 1 in Ref. [2], we considered a classical one-bit memory. Let \( t (=0 < t < 1) \) be the volume fraction of the left box corresponding to “0,” and \( p \) the probability that the memory registers outcome “0” in its initial state. The initial state is not in equilibrium except for the case of \( p = 1 \). According to the standard probability theory [3,4], the total entropy (SE or VNE) of the memory, \( S^M \), can be decomposed as \( S^M = \rho S^M_0 + (1 - \rho) S^M_1 + H(p) \), where \( S^M_k \) is the entropy of the conditional state that registers “\( k \)” \((=0, 1)\), and \( H(p) \equiv -p \ln p - (1 - p) \ln(1 - p) \) is the Shannon information stored in the memory. We assume that the conditional states are in thermal equilibrium. Therefore we can define the free energies corresponding to the conditional equilibrium states as \( F^M_k = -\beta^{-1} S^M_k \) because of the internal-energy degeneracy between “0” and “1.”

To erase the information, we reset the memory state to the standard state 0 with unit probability [5,6]. After the erasure, the probability of the memory being found in state 0 is unity so that \( \Delta S^M = S^M_0 - [\rho S^M_0 + (1 - \rho) S^M_1 + H(p)] \). Applying inequality (1), we reproduce our main result in Ref. [2]: \( W_{\text{erasure}} = -Q \approx -\Delta S^M = \beta^{-1} H(p) = (1 - p)\langle F^M_1 - F^M_0 \rangle \), where \( (1 - p)\langle F^M_1 - F^M_0 \rangle = \langle [\ln p - \ln(1 - p)] \rangle \) in the present situation. The error of Dillenschneider and Lutz lies in their assumption \( \Delta S = -\sum_n p_n \ln p_n \), which is valid only for the special case of \( t = 1/2 \). We also note that our bound differs from Landauer’s by the difference of the averaged conditional free energies: \( (1 - p)\langle F^M_1 - F^M_0 \rangle = [p F^M_0 + (1 - p) F^M_1] - F^M_0 \).

For symmetric memories, the amount of a decrease in \( H(p) \) should be dissipated into the environment as the heat, as is the case for the original Landauer’s principle, because \( S^M_0 = S^M_1 \) so that \( \Delta S^M = -H(p) \geq \beta^{-1} Q \) holds. On the other hand, for asymmetric memories, the decrease can be compensated for by an increase in the entropy \( S^M_0 - [\rho S^M_0 + (1 - \rho) S^M_1] \), so that \( \Delta S^M = 0 \) can hold. Thus, although Landauer’s principle breaks down, the second law of thermodynamics remains intact.

Dillenschneider and Lutz also state that “the entropy, both before and after erasure, is indeed invariant under \( t \rightleftharpoons 1 - t \); so should the work \( W_{\text{erasure}} \).” While our choice of the standard state is arbitrary (0 or 1), the standard state needs to be predetermined to make the erasure process well defined, because by definition the initialization of the memory requires to return to the fixed standard state [5,6]. Therefore, entropy \( S^M \) after erasure is not invariant under \( t \rightleftharpoons 1 - t \) for asymmetric memories.

Finally, we prove (1) for quantum cases. Let \( \hat{\rho}(t) \) be the density operator of the total system at time \( t \) consisting of memory \( M \) and heat bath \( B \). We define \( \hat{\rho}^M(t) = \text{tr}_B[\hat{\rho}(t)] \) and \( \hat{\rho}^B(t) = \text{tr}_M[\hat{\rho}(t)] \), and \( \hat{\rho}^\text{can} = e^{-\hat{H}^B/\beta} / \text{tr}(e^{-\hat{H}^B/\beta}) \) with \( \hat{H}^B \) being the Hamiltonian of the heat bath. We consider a unitary evolution of the total system from time 0 to \( t \). Let \( S(\hat{\rho}) = -\text{tr}(\hat{\rho} \ln \hat{\rho}) \) be the VNE of \( \hat{\rho} \). We assume that \( \hat{\rho}(0) = \hat{\rho}^M(0) \otimes \hat{\rho}^\text{can} \). Then we have \( S(\hat{\rho}^M(0)) + S(\hat{\rho}^\text{can}) = S(\hat{\rho}(0)) \leq S(\hat{\rho}^M(t)) + S(\hat{\rho}^B(t)) \leq S(\hat{\rho}(t)) - \text{tr}(\hat{\rho}^B(t) \ln \hat{\rho}^\text{can}) \), where we used the convexity of VNE and Klein’s inequality. Therefore, \( S(\hat{\rho}^M(t)) - S(\hat{\rho}(0)) \geq \beta t[H^B(\hat{\rho}^M(0) - \hat{\rho}^B(t))] \) holds, which proves (1).

Takahiro Sagawa and Masahito Ueda

1Department of Physics, University of Tokyo
7-3-1, Hongo, Bunkyo-ku, Tokyo, 113-8654, Japan
2ERATO Macroscopic Quantum Control Project, JST
2-11-16 Yayoi, Bunkyo-ku, Tokyo 113-8656, Japan

Received 2 October 2009; published 14 May 2010
DOI: 10.1103/PhysRevLett.104.198904
PACS numbers: 05.70.Ln, 03.67.-a, 05.30.-d