Stabilization of a matter-wave droplet in free space by feedback control of interatomic interactions

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A Bose-Einstein condensate in three-dimensional free space is known to be unstable against expansion or collapse if the scattering length of atoms is constant. We propose a method to produce a stable self-trapped condensate in which nondestructive measurement of the condensate’s peak density is used to perform feedback control of the scattering length. The stability is shown to be robust against experimental imperfections in the measurements.

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I. INTRODUCTION

Matter-wave bright solitons of a Bose-Einstein condensate (BEC) in a quasi-one dimensional (quasi-1D) trapping potential have been realized by the ENS group [1] and the Rice group [2]. The stability of this self-trapped state is achieved by the balance between the attractive interatomic interaction and the quantum kinetic pressure. In 1D, the bright soliton is stable and robust against noise. In 2D and 3D, however, the balance between the attractive interaction and the kinetic pressure is precarious, and small deviations from the stationary state cause a collapse or expansion of the system. While self-trapped liquids, such as raindrops, are quite common, a self-trapped gas is a novel state of matter. Such a stable self-trapped state in a gaseous BEC in 2D or 3D, if it realized, might be referred to as a matter-wave drop.

A scheme proposed in Refs. [3,4] to stabilize a BEC droplet involves the rapid oscillation of the interatomic interaction using Feshbach resonance [5,6]. The rapid oscillation of the interaction produces an effective potential that prevents the condensate from collapsing. This phenomenon is analogous to the stabilization of an inverted pendulum by an oscillating pivot [7]. Several researchers have demonstrated the stabilization of a BEC droplet in 2D with oscillating interactions by numerically solving the Gross-Pitaevskii (GP) equation [3,4,8,9]. Montesinos et al. [10] have shown that this stabilization in 2D is also possible for a multicomponent BEC. Matuszewski et al. [11] have predicted 3D breather solitons confined in a 1D lattice. However, in 3D free space, it appears that an oscillating interaction alone cannot stabilize a BEC droplet due to its dynamical instabilities [4,8]. By taking into account the effect of energy dissipation, which is usually present in actual situations, we have shown that a BEC droplet with oscillating interactions can be stabilized in 3D [12].

In the present paper, we show that a BEC droplet in 3D free space can be stabilized by feedback control of the interatomic interaction through nondestructive measurement of the condensate’s peak column density. Real-time monitoring of the density profile of a condensate is possible by using a nondestructive in situ imaging method [13]. The collapse or expansion of the condensate can be preempted by a decrease or an increase, respectively, in the strength of the attractive interaction if the peak density of the condensate increases above or decreases below a prescribed value. Thus, the BEC droplet can be stabilized without collapse or expansion by a negative feedback control of the interaction through the real-time monitoring of the condensate’s peak density. We will also examine the effect of experimental imperfections in real-time measurement, such as spatial and temporal resolutions and experimental errors, and show that the stabilized BEC droplet is robust against these imperfections.

This paper is organized as follows. Section II discusses the stationary solutions of the GP equations in 1D, 2D, and 3D free space. Section III numerically investigates the dynamics of the condensate under the negative feedback control and shows that a BEC droplet in 3D can be stabilized for a wide range of parameters. Section IV presents a variational analysis using a Gaussian trial function. Section V studies the stability against measurement errors and finite measurement resolution and shows that a BEC droplet is robust against these experimental imperfections. Finally, Sec. VI provides a conclusion.

II. STATIONARY SOLUTIONS OF THE GROSS-PITAEVSKII EQUATION IN FREE SPACE

We first consider the stationary states of a BEC with an attractive interaction in 1D, 2D, and 3D free space. The dynamics of the BEC is described by the GP equation

\[
\frac{i\hbar}{2m} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi + \frac{4\pi\hbar^2 a}{m} |\psi|^2 \psi,
\]

where \( m \) is the mass of an atom, \( V \) is an external potential, and \( a \) is the s-wave scattering length. The wave function is normalized as \( \int |\psi|^2 \, dr = N \), with \( N \) being the number of atoms.

When the external potential is given by \( V=m\omega_d^2(x^2+y^2)/2 \) with \( \hbar\omega_d \) being much larger than the other characteristic energies, the degrees of freedom of the BEC in the \( x \) and \( y \) directions are frozen and the system behaves effectively as a 1D system. Writing the wave function as...
\[ \psi(r) = \left( \frac{m \omega_{2d} \mu}{\pi \hbar} \right)^{1/4} e^{-\left( m \omega_{2d} \mu z^2 / 2 \right)} e^{i \omega_{2d} \mu z / 2} \psi_{s}(x,y) \]

and integrating Eq. (1) over \( x \) and \( y \), we obtain an effective 1D equation

\[ i \hbar \frac{\partial \psi_z}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_z}{\partial z^2} + g_{1d} |\psi_z|^2 \psi_z, \tag{3} \]

where \( g_{1d} = 2 \hbar \omega_{1d} a \). If \( a < 0 \), this equation has a well-known soliton solution \[ \psi_z = e^{-i \mu \mu \hbar} \sqrt{\frac{\eta N}{2}} \text{sech} \left( \eta(z - z_0) \right), \tag{4} \]

where \( \mu = -m \omega_{1d} a^2 N^2 / 2 \), \( \eta = m \omega_{1d} a |N| / \hbar \), and \( z_0 \) is the peak position of the soliton. Equation (4) is the ground-state solution free from dynamical instabilities. Thus an attractive 1D BEC is stable.

When \( V = m \omega_{2d} a^2 / 2 \) and \( \hbar \omega_{2d} \) is much larger than the other characteristic energies, the system behaves effectively as a 2D system. By substituting the wave function

\[ \psi(r) = \left( \frac{m \omega_{2d} \mu}{\pi \hbar} \right)^{1/4} e^{-\left( m \omega_{2d} \mu a^2 / 2 \right)} e^{i \omega_{2d} \mu a z / 2} \psi_{s}(x,y) \]

in Eq. (1) and integrating the result over \( z \), we obtain an effective 2D equation

\[ i \hbar \frac{\partial \psi_{s}}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi_{s}}{\partial x^2} + \frac{\partial^2 \psi_{s}}{\partial y^2} \right) + g_{2d} |\psi_{s}|^2 \psi_{s}, \tag{6} \]

where \( g_{2d} = (8 \pi N^3 \alpha^2 \omega_{2d} a / m)^{1/2} \). At the critical strength of the interaction

\[ g_{2d} = 1.862 \frac{\pi \hbar^2}{mN}, \tag{7} \]

Eq. (6) has a stationary self-trapped solution, which is known as the Townes soliton [15]. If the peak of the Townes soliton is located at the origin \( r = 0 \), the wave function is axisymmetric and the density \( |\psi|^2 \) monotonically decreases to zero for \( r \rightarrow \infty \). The Townes soliton also has a scaling property; if \( \psi_{s}(x,y) \) is a stationary solution of Eq. (6), the scaled wave function \( \alpha \psi_{s}(\alpha x, \alpha y) \) with an arbitrary scaling parameter \( \alpha \) is also a stationary solution of Eq. (6). However, the Townes soliton is dynamically unstable because an infinitesimal deviation from the stationary solution grows exponentially in time, making the Townes soliton eventually collapse or expand. Thus, an attractive 2D BEC is marginally stable at a particular interaction strength (7).

When \( V = 0 \)—i.e., in 3D free space—Eq. (1) with \( a < 0 \) has a stationary self-trapped solution that is dynamically unstable like the Townes soliton. The density of this stationary solution has spherical symmetry and monotonically decreases to zero for \( r \rightarrow \infty \) [16]. The striking difference between this 3D stationary state and the Townes soliton is exhibited in their scaling properties. Normalizing the length, time, and wave function in units of \( \ell, m \ell^2 / \hbar \), and \( (N / \ell^3)^{1/2} \), respectively, where \( \ell \) is an arbitrary length scale, Eq. (1) reduces to

\[ \psi(r) = \sqrt{\frac{m \omega_{1d} \mu}{\pi \hbar}} e^{-\left( m \omega_{1d} \mu (r^2 + y^2) / 2 \right)} e^{i \omega_{1d} \mu r / \ell} \]

where \( g_{1d} = 2 \hbar \omega_{1d} a \). If \( a < 0 \), this equation has a well-known soliton solution [14]

\[ \psi_z = e^{-i \mu \mu \hbar} \sqrt{\frac{\eta N}{2}} \text{sech} \left( \eta(z - z_0) \right), \]

\[ \psi_{s} = \sqrt{\frac{m \omega_{2d} \mu}{\pi \hbar}} e^{-\left( m \omega_{2d} \mu a^2 z^2 / 2 \right)} e^{i \omega_{2d} \mu a z / \ell} \]

\[ \frac{\partial \psi_{s}}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi_{s}}{\partial x^2} + \frac{\partial^2 \psi_{s}}{\partial y^2} \right) + g_{2d} |\psi_{s}|^2 \psi_{s}, \]

\[ g_{2d} = 1.862 \frac{\pi \hbar^2}{mN}, \]

\[ |\psi|^2 \]

\[ \frac{\partial |\psi|^2}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 |\psi|^2}{\partial x^2} + \frac{\partial^2 |\psi|^2}{\partial y^2} \right) + g_{2d} |\psi|^2 \psi, \]

\[ g_{2d} = 1.862 \frac{\pi \hbar^2}{mN}, \]

\[ \psi(r) = \sqrt{\frac{m \omega_{1d} \mu}{\pi \hbar}} e^{-\left( m \omega_{1d} \mu (r^2 + y^2) / 2 \right)} e^{i \omega_{1d} \mu r / \ell} \]

where \( g = 4 \pi N a / \ell \). Therefore, if \( \psi(r,t) \) is a solution of the GP equation with scattering length \( a \), the scaled wave function \( \alpha^{3/2} \psi(\alpha r, \alpha^2 t) \) also satisfies the GP equation with scattering length \( a / \alpha \). This indicates that there always exists an unstable stationary solution for any \( g < 0 \) and the solutions for different \( g \)'s are related by the scaling transformation. This suggests the possibility that we can stabilize the 3D matter-wave droplet for any value of \( g < 0 \) if we are able to keep the droplet in the corresponding stationary state. This can indeed be done by way of feedback control as discussed in the next section.

In Fig. 1, we plot the density profile of the unstable stationary state in 3D for \( g = -(2 \pi)^{3/2} \), which is numerically obtained by the Newton-Raphson method [17]. The inset of Fig. 1 shows the logarithmic plot of the large-\( r \) behavior. We find that the tail of the unstable stationary state (solid curve) is much longer than that of the Gaussian wave function \( e^{-r^2 / \pi^{3/2}} \) (dashed curve). Generally, in \( d \) dimensions the unstable stationary wave function has an asymptotic form \( r^{(1-d)/2} e^{-r^2} \) with constant \( c \) for \( r \rightarrow \infty \) [16]. The dependence of the tail on \( r \) in the inset of Fig. 1 is consistent with this functional form with \( d = 3 \) and \( c \approx 1.3 \).

\[ \int_{0}^{\infty} d^3 r \psi^2 = \frac{\pi \hbar^2}{mN}, \]

\[ \psi(r) = \sqrt{\frac{m \omega_{1d} \mu}{\pi \hbar}} e^{-\left( m \omega_{1d} \mu (r^2 + y^2) / 2 \right)} e^{i \omega_{1d} \mu r / \ell} \]

where \( g = 4 \pi N a / \ell \). Therefore, if \( \psi(r,t) \) is a solution of the GP equation with scattering length \( a \), the scaled wave function \( \alpha^{3/2} \psi(\alpha r, \alpha^2 t) \) also satisfies the GP equation with scattering length \( a / \alpha \). This indicates that there always exists an unstable stationary solution for any \( g < 0 \) and the solutions for different \( g \)'s are related by the scaling transformation. This suggests the possibility that we can stabilize the 3D matter-wave droplet for any value of \( g < 0 \) if we are able to keep the droplet in the corresponding stationary state. This can indeed be done by way of feedback control as discussed in the next section.

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### III. FEEDBACK CONTROL OF A BEC DROPLET

The 3D stationary solution of Eq. (8) is dynamically unstable against collapse or expansion. The aim of the present paper is to show that we can stabilize it by a feedback control of the scattering length \( a \). When the system is about to collapse, we can increase \( a \) to prevent the collapse. Similarly, a decrease in \( a \) can prevent expansion. Thus, by measuring the density profile of the condensate in a nondestructive manner, we can achieve the feedback control of \( a \) to stabilize the BEC droplet.
An observable quantity in nondestructive phase-contrast imaging [13] is the column density of the condensate, given by
\[ d_{\text{col}}(x,y,t) = \int dz|\phi(r,t)|^2, \] (9)

where the line of sight is assumed to be in the \( z \) direction. We use the peak value of the column density,
\[ D = d_{\text{col}}(0,0,t), \] (10)

for the feedback control.

Let \( D_0 \) be a target value of the stabilized column density. The feedback loop should operate on the strength of the interaction \( g \) in such a manner that any deviation \( D-D_0=\tilde{D} \) from the target value \( D_0 \) will be suppressed. We assume that the feedback function has the form
\[ \dot{g} = A(D-D_0) + BD + CD\dot{D}, \] (11)
where \( A, B, \) and \( C \) are dimensionless constants. The first term on the right-hand side drives \( D \) to \( D_0 \), and the time derivatives of \( D \) in the second and third terms work to suppress the overshoot. The asymptotic value of \( g = g_0 \) is not predetermined, since the number of atoms in the condensate may decrease during time evolution for lack of the trapping potential. This is the reason why the left-hand side of Eq. (11) is not \( g-g_0 \) but \( \dot{g} \). Recovering the dimensions of \( D \) and \( t \), we can write Eq. (11) as
\[ \dot{g} = \frac{\ell^3}{4\pi N^2 m} A(D-D_0) + \frac{\ell^5}{4\pi N^2 m} B \dot{D} + \frac{m\ell^5}{4\pi N^2 h} C \ddot{D}, \] (12)

where \( \ell \) is an arbitrary unit of length used to obtain Eq. (8).

In experiments, phase-contrast images are taken by a charge-coupled-device (CCD) camera at a certain frame rate (e.g., 20 kHz in Ref. [18]). The measurement of \( D \) is performed at discrete times, and the time derivatives in Eq. (11) are approximately obtained by
\[ \dot{D}(t) = \frac{1}{\Delta t} [D(t) - D(t - \Delta t)], \] (13a)
\[ \ddot{D}(t) = \frac{1}{\Delta t^2} [D(t) - 2D(t - \Delta t) + D(t - 2\Delta t)], \] (13b)

where \( \Delta t \) is the interval between measurements. Accordingly, \( g \) is also changed stepwise as
\[ g(t) = g(t - \Delta t) + \Delta t A[D(t) - D_0] + B[D(t) - D(t - \Delta t)] + \frac{C}{\Delta t^2} [D(t) - 2D(t - \Delta t) + D(t - 2\Delta t)]. \] (14)

The interval \( \Delta t \) must be made much smaller than the characteristic time scale of the dynamics. The \( \Delta t \) dependence of the stability is discussed in Sec. V.

We assume that the initial state is the noninteracting ground state in an isotropic harmonic potential \( V=\hbar \omega r^2/2 \). Henceforth, we take the units of length and time to be \( \ell = [\hbar/(m\omega)]^{1/2} \) and \( \omega^{-1} \), respectively. The initial state is then described by the Gaussian wave function \( \psi=e^{-r^2/2}/\pi^{1/4} \). At

\[ t=0, \] the trapping potential is suddenly switched off and the strength of the interaction is set to be \( g=-(2\pi)^{3/2} \). The strength of the interaction \( g \) evolves in time according to Eq. (14), where the value of \( D_0 \) is chosen to be the peak column density of the initial wave function, \( D_0=\int dz |\psi(x=0,y=0,z,t=0)|^2=1/\pi \). The wave function evolves in time according to Eq. (8), which is numerically solved by the Crank-Nicolson method. We neglect the effect of gravity by assuming that it is canceled using, e.g., a technique of magnetic levitation [19].

Figure 2(a) shows the time evolution of \( D \) for \( A=10, B=100, C=50, \) and \( \Delta t=0.01 \). The value of \( D \) deviates significantly from \( D_0 \) only for \( t\leq 20 \) and quickly converges to \( D_0 \) thereafter. This indicates that the feedback control of the interaction successfully works to stabilize a BEC droplet in 3D. The inset in Fig. 2(b) shows the density profile of the stabilized condensate at \( t=100 \). This converged density profile is found to be related to the stationary solution of Eq. (8) by appropriate scaling. The feedback control can thus transform a Gaussian wave function into a stationary solution of Eq. (8). During this transformation, a certain fraction of atoms are lost from the condensate. The solid curve in Fig.
shows the fraction of atoms remaining within the radius \( r = 10 \) around the center of the condensate,

\[
\frac{N}{N_0} = \int_0^{10} \, dr \, 4 \pi r^2 |\psi|^2 ,
\]

We find that roughly 10% of the atoms are scattered away from the condensate without returning to the center for lack of the trapping potential.

Figure 3 shows the dependence of the dynamics on the feedback parameters. Figure 3(a) shows that the parameter \( A \) controls the amplitude of density oscillations. The oscillation frequency increases with an increase in \( A \), and for large \( A \) the system becomes unstable against the growth in the amplitude of the density oscillations. This is because the term of \( A \) in Eq. (11) plays a role of pulling the value of \( D \) back to \( D_0 \). If \( A \) is too large, this pulling causes an overshoot, resulting in oscillations of \( D \). Figures 3(b) and 3(c) indicate that the amplitudes of the oscillations decrease with increasing \( B \) and \( C \), and the \( B \) and \( C \) terms in Eq. (11) work to suppress the overshoot. The decay of the oscillations is faster for larger \( B \) Fig. 3(b) and slower for larger \( C \) Fig. 3(c).

Figure 4 shows a stability diagram of the feedback control for \( A = 10 \) with respect to the feedback parameters \( B \) and \( C \). We find that the BEC droplet can be stabilized for \( B \approx 30 \) and \( C \approx -20 \).

IV. VARIATIONAL ANALYSIS

In order to understand in an analytic manner the stabilization of the system due to the feedback control, we conduct a variational analysis. We employ a Gaussian trial function

\[
\psi_G = \frac{1}{\pi^{3/4} R^{3/2}} e^{-r^2/2 + (R^2/2R^2) r^2} ,
\]

where \( R(t) \) is the variational parameter that characterizes the size of the condensate. Here, for simplicity, we neglect the atomic loss shown in Fig. 2(b).

Equation (8) is derived by the application of the variational principle to the action

\[
S = \int dt \, \rho^* \left( \frac{\partial}{\partial t} + \mathbf{\nabla}^2 - \frac{g}{2} |\psi|^2 \right) \psi .
\]

Substituting the variational wave function (16) into Eq. (17) and minimizing the result with respect to \( R \), we obtain the equation of motion for \( R \) as

\[
\dot{R} = - \frac{1}{R^3} - \frac{g}{(2 \pi)^{3/2}} \frac{1}{R^4}.
\]

The unstable stationary solution of Eq. (18) is obtained for \( R = -g/(2 \pi)^{3/2} \); hence, if \( g = -(2 \pi)^{3/2} \), we have \( R = 1 \). In Fig.
where we have kept only the terms linear in \( \tilde{D} \). Comparing it with the numerically obtained stationary solution of Eq. (8) with \( g = -(2 \pi)^{3/2} \) (solid curve), we see that the central density is larger and the tail is longer for the stationary solution than for the Gaussian function.

Since \( R \) cannot be directly measured in experiments, we rewrite the equation of motion in terms of the central column density

\[
D = 2 \int_0^\infty dr |\psi(r)|^2 = \frac{1}{\pi R^2}.
\]

We consider small deviations from \( g_0 = -(2 \pi)^{3/2} \) and \( R_0 = 1 \), and decompose \( D \) and \( g \) as \( D = D_0 + \tilde{D} \) and \( g = g_0 + \tilde{g} \) with \( D_0 = 1/\pi \). Equation (18) can then be rewritten as

\[
\dot{\tilde{g}} = A \tilde{D} + B \dot{\tilde{D}} + CD \tilde{D}.
\]

Differentiating Eq. (20) with respect to \( t \) and using Eq. (21), we obtain

\[
\frac{d}{dt} \begin{pmatrix} \tilde{D} \\ \dot{\tilde{D}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -A/\sqrt{2 \pi^{3/2}} & 1 - B/\sqrt{2 \pi^{3/2}} & -C/\sqrt{2 \pi^{3/2}} \end{pmatrix} \begin{pmatrix} \tilde{D} \\ \dot{\tilde{D}} \end{pmatrix}.
\]

If the \( 3 \times 3 \) matrix in Eq. (22) has an eigenvalue whose real part is positive, \( \tilde{D} \) diverges exponentially in time. Therefore, the condition for stability is that all the eigenvalues have negative real parts, which can be examined by using the Routh-Hurwitz criterion [21] without actually solving the eigenvalue equation. The stability condition of Eq. (22) is found to be (see the Appendix for derivation)

\[
A > 0, \\
B > \sqrt{2 \pi^{3/2}}, \\
C > 0, \\
A < C \left( \frac{B}{\sqrt{2 \pi^{3/2}}} - 1 \right).
\]

These inequalities qualitatively agree with the numerically obtained stability diagram in Fig. 4. The critical value of \( B \) in Eq. (23b) is \( \sqrt{2 \pi^{3/2}} \approx 24.7 \), which is in good agreement with the numerically obtained value \( \sim 30 \). The critical value of \( C \) in Eq. (23c) is zero, while Fig. 4 shows that it is negative, \( \sim -20 \). Thus, while the Gaussian approximation requires that the \( C \) term in Eq. (11) be present for the stabilization of the BEC droplet, actually the \( C \) term is not crucial for the stabilization.

In Fig. 5, we plot the Gaussian function solution of Eq. (1) with \( \epsilon = 0 \) (solid line), \( \epsilon = 0.2 \) (dashed line), \( \epsilon = 0.6 \) (dotted line), and \( \epsilon = 0.7 \) (dotted-dashed line). The feedback parameters are \( A = 10 \), \( B = 100 \), and \( C = 0 \). The other conditions are the same as those in Fig. 2.

V. EFFECTS OF EXPERIMENTAL IMPERFECTIONS ON THE STABILITY OF FEEDBACK CONTROL

We have thus far taken for granted that the peak column density \( D \) can be measured precisely. However, in actual experiments, the measurement process always contains imperfections, such as errors and the finite resolution of time and space. In this section, we investigate the effects of such imperfections on the stability of our feedback control.

We first study the effects of measurement errors of \( D \) on the stability. We assume that a measurement outcome which contains an error is given by

\[
D_{\epsilon} = D(1 + \epsilon v_{\text{err}}),
\]

where \( D \) is defined by Eq. (10), \( \epsilon \) describes the error level, and \( v_{\text{err}} \) is a random variable that simulates the measurement error and is assumed to obey the normal distribution \( e^{-v_{\text{err}}^2/2}/\sqrt{2 \pi} \). Figure 5 shows some examples of the time evolution of \( D_{\epsilon} \), in which \( D_{\epsilon} \) is used in the feedback equation (11) instead of \( D \). We find that the system is tolerant against the error level up to about 60% in every measurement and the level of stability is quite high. In Fig. 5, we see that \( D \) has a tendency to decrease with an increase in \( \epsilon \). This phenomenon is similar to that in the case of oscillating interactions [3,4], where the peak density is suppressed by the oscillation of the interaction. In the present case, the random fluctuations in \( g \) play the role of oscillating interactions.

We next study the effect on the stability of the spatial resolution in the measurement of \( D \). We assume that due to the finite spatial resolution, the measured value is filtered by a Gaussian function

\[
D_{\sigma} = \int dx dy \frac{1}{2 \pi \sigma^2} e^{-(x^2+y^2)/2\sigma^2} d_{\text{col}}(x,y),
\]

where \( d_{\text{col}}(x,y) \) is given in Eq. (9) and \( \sigma \) characterizes the spatial resolution. Figure 6 shows the time evolution of \( D \), in which \( D_{\sigma} \) is used in Eq. (11) instead of \( D \). It is remarkable that the stability is very robust against finite spatial resolu-
Then, the resolution of the phase-contrast imaging must be relaxed.

We use larger condensates atoms and take the units of length and time to be 3.5 μm and 16 ms, respectively, which correspond to \( \omega = 10 \times 2\pi \) Hz. Then, the resolution of the phase-contrast imaging must be \( \sigma \leq 1.7 \) μm and the interval of the measurements must be \( \Delta \tau \approx 6.4 \) ms, which corresponds to a frame rate \( \approx 160 \) Hz. If we use larger condensates (i.e., smaller \( \omega \)), these restrictions can be relaxed.

VI. CONCLUSIONS

We have shown that a BEC droplet (self-trapped condensate) can be stabilized in 3D free space by the feedback control of the strength of the interaction between atoms. By negative feedback on the strength of the interaction through nondestructive monitoring of the condensate’s peak column density, we can prevent the condensate from collapsing and expanding. Even starting from a Gaussian wave function, the system can reach the stationary state of the GP equation by feedback control.

We have considered the feedback from the peak column density \( D \) and its time derivatives \( \dot{D} \) and \( \ddot{D} \) [Eq. (11)] and have examined the stability of the system for various parameter values and shown that stability is achieved for a wide range of parameters, as demonstrated in Figs. 3 and 4.

We have also investigated stability against experimental imperfections, such as measurement errors (Fig. 5), finite spatial resolution (Fig. 6), and finite time intervals between measurements. We have found that the stability is very robust against these imperfections.

In the present paper, we have considered only the simplest form of negative feedback. More robust stability should be achieved and the stationary state can be reached more quickly if more sophisticated methods are used, such as the Kalman filter [22] and robust control [23].

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APPENDIX: STABILITY CONDITION WITH THE ROUGHT-HURWITZ CRITERION

According to the Routh-Hurwitz criterion [21], all solutions of a polynomial equation

\[
a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n = 0
\]

have negative real parts if (i) all the coefficients \( a_i \) for \( i = 0, 1, \ldots, n \) are real and positive and (ii) the determinants

\[
\begin{vmatrix}
a_1 & a_3 & a_5 & \cdots & a_{2i-1} \\
a_0 & a_2 & a_4 & \cdots & a_{2i-2} \\
0 & a_4 & a_3 & \cdots & a_{2i-3} \\
0 & a_0 & a_2 & \cdots & a_{2i-4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & a_i \\
\end{vmatrix}
\]

for \( i = 2, 3, \ldots, n \) are positive.

The system described by Eq. (22) is stable if all the eigenvalues of the 3 × 3 matrix on the right-hand side have negative real parts. The eigenvalue equation is given by

\[
\lambda^3 + \frac{C}{\sqrt{2} \pi^{3/2}} \lambda^2 + \left( \frac{B}{\sqrt{2} \pi^{3/2}} - 1 \right) \lambda + \frac{A}{\sqrt{2} \pi^{3/2}} = 0.
\]

Applying the Routh-Hurwitz criterion to Eq. (A3), we obtain the condition for the stability given in Eq. (23).