Quantum theory features two types of uncertainty: indeterminacy of observables and complementarity of quantum measurements. The indeterminacy [1] reflects the inherent nature of a quantum system [2–4], whereas the complementarity [5] involves quantum measurement, and estimation of a quantum state from the measurement outcomes is essential [6–8]. However, how to optimize the measurement and estimation for a given quantum system has remained an outstanding issue. The purpose of this paper is to report the resolution to this problem.

The complementarity implies that we cannot simultaneously perform precise measurements of noncommuting observables. There must exist trade-off relations between the measurement errors of the noncommuting observables. Whereas a number of trade-off relations have been found, they are neither achievable for all quantum states and observables [8–12] nor applicable to all quantum systems [13–15]. Due to advances in controlling quantum states, it is now possible to implement a scheme that performs a projection measurement on a part of samples and another projection measurement on the rest [16–19]. However, the achievable bound of the measurement errors for such a scheme is yet to be identified.

In this paper, we report the following three results. First, we prove that for all measurements the measurement errors of noncommuting observables are bounded from below by their commutation relation. This implies that not only quantum fluctuations but also measurement errors are bounded by the same commutation relation. However, the bound cannot be achieved for all quantum states and observables. Second, we find the achievable bound for the measurements that perform a projection measurement with or without noise on a part of samples and another measurement on the rest. We propose a scheme of the experimental setup that achieves the bound. Third, we numerically show that the bound is satisfied for all measurements. Therefore, we conjecture that all measurements satisfy the proposed trade-off relation, and that the measurements that achieve the bound are optimal for obtaining information about two noncommuting observables.

II. QUANTUM ESTIMATION THEORY OF ERROR IN MEASUREMENT

A. Definition of error in measurement

Given \( n \) independent and identically distributed (i.i.d.) unknown quantum states \( \beta \) on the \( d \)-dimensional Hilbert space
where the summation is taken over all non-negative integers of the estimator $\hat{\lambda}$ operator valued measure (POVM) measurement $M$.

we assume that the space of the possible quantum states is

measurements and the estimation process, is calculated to be

the outcome $p_i$ is the probability that the outcome $\hat{\lambda}_i$ corresponding to the observable $\hat{X}$ is obtained ($\sum_i n_i = n$). The expectation value $E[X^{\text{est}}]$ of the estimator $X^{\text{est}}$ is equivalent to $\langle \hat{X} \rangle$,

where the summation is taken over all non-negative integers $n_i \geq 0$ that satisfy $\sum_i n_i = n$, and

is the probability that the outcome $i$ is obtained $n_i$ times with $p_i = \text{Tr}[\hat{\rho} \hat{P}_i]$. The estimators, such as (1), that satisfy (2) for all quantum states are called unbiased estimators. The variance of the estimator $X^{\text{est}}$, which quantifies the error in the overall measurements and the estimation process, is calculated to be characterizes the quantum fluctuation of observable $\hat{X}$. Since the variance decreases as $n^{-1}$, we can estimate $\langle \hat{X} \rangle$ from the measurement outcomes of $P$ if $n$ is sufficiently large.

When we perform the positive-operator-valued measure (POVM) measurement $M = \{M_i\}_{i=1}^m$, the estimator cannot, in general, be written in the form of (1), and the variance is asymptotically greater than that of the optimal one:

where the left-hand side (LHS) and the right-hand side (RHS) show the variance of the concerned estimator and that of the optimal case per sample, respectively. The variance $\text{Var}[X^{\text{est}}]$ is caused by three different types of errors: quantum fluctuations, errors in the $n$ identical measurements, and errors in the estimation process (see Fig. 1). The estimation error arises unless we use optimal estimators that minimize $\text{Var}[X^{\text{est}}]$ such as the maximum likelihood estimator. To quantify the error in single-shot measurements, it is necessary to use the estimator that minimizes the variance. We define the measurement error as

where the minimization is taken over all consistent estimators $\{\hat{\lambda}_i\}_{i=1}^d$ that asymptotically converge to $\langle \hat{X} \rangle$:

for all quantum states $\hat{\rho}$ and arbitrary $\delta > 0$. The condition (8) implies that the POVM measurement does not involve any systematic error and we can quantify the measurement precisely. Examples of the consistent estimator include the average of eigenvalues (1) for the projection measurement, and the maximum likelihood estimator for the POVM measurement. These quantities also minimize $\lim_{n \to \infty} n \text{Var}[X^{\text{est}}]$. If there exists no consistent estimator of $\langle \hat{X} \rangle$, we define $\epsilon(\hat{X}; M) = +\infty$. Such a situation occurs, for example, when the projection measurement of an observable that does not commute with $\hat{X}$ is performed.

We note that the measurement error $\epsilon(\hat{X}; M)$ is defined as the limit of $n$ going to infinity. If $n$ is finite and not sufficiently large, we cannot use the statistical analysis to evaluate the errors. Therefore, the definition (7) is meaningful for the case in which $n$ is infinite or, at least, sufficiently large.

B. Measurement error in terms of Fisher information

To express $\epsilon(\hat{X}; M)$ in terms of the Fisher information, we expand the Hermitian operators on the $d$-dimensional Hilbert space by the generators of the Lie algebra $\text{su}(d)$. The generators $\hat{\lambda}_i = \{\hat{\lambda}_i\}_{i=1}^{d^2-1}$ are traceless, Hermitian, and orthonormal with respect to the trace-norm:

In terms of them, an arbitrary quantum state $\hat{\rho}$ can be expanded as

$\hat{\rho} = \frac{1}{d} \hat{I} + \theta \cdot \hat{\lambda} = \frac{1}{d} \hat{I} + \sum_{i=1}^{d^2-1} \theta_i \hat{\lambda}_i$. (10)
where \( I \) is the identity operator and \( \theta \in \mathbb{R}^{d^2-1} \) is a \((d^2-1)\)-dimensional real vector. Since \( \hat{\rho} \) is unknown, \( \theta \) is unknown, and the dimension of the space \( \Theta \) of all possible parameters \( \theta \) is \( d^2-1 \). An arbitrary observable can also be expanded in terms of the same set of generators as
\[
\hat{X} = x_0 I + x \cdot \hat{\lambda}.
\]
(11)
Then, the expectation value can be evaluated as
\[
\langle \hat{X} \rangle = x_0 + x \cdot \theta.
\]
(12)
Therefore, estimating \( \langle \hat{X} \rangle \) amounts to estimating \( x \cdot \theta \). For any consistent estimator \( X^\text{est} \) of \( \langle \hat{X} \rangle \), the variance \( \text{Var}[X^\text{est}] \) satisfies the following Cramér-Rao inequality [22]:
\[
\lim_{n \to \infty} n \text{Var}[X^\text{est}] \geq \begin{cases} 
\frac{1}{\text{det}(J(M))} & x = \text{supp}(J(M)), \\
+\infty & \text{otherwise}, 
\end{cases}
\]
(13)
where \( J(M) \) is the Fisher information matrix whose \( ij \) component is defined as
\[
[J(M)]_{ij} := \sum_k p_k (\partial_i \ln p_k)(\partial_j \ln p_k),
\]
(14)
where \( \partial_i = \partial/\partial \theta_i \) and \( p_k = \text{Tr}[\hat{\rho} \hat{M}_k] \). In general, \( J(M) \) does not have the inverse. The inverse in (13) is defined by the Moore-Penrose pseudoinverse [23,24]. In the rest of this paper, the inverses of nonsquare matrices and matrices that have zero eigenvalue are defined by the Moore-Penrose pseudoinverse. If the RHS of (13) is finite, there exists some estimator, for example, the maximum likelihood estimator, that achieves the equality of (13). If the RHS of (13) is infinite, there exists no consistent estimator of \( \langle \hat{X} \rangle \).

Therefore, the measurement error can be written as
\[
\epsilon(\hat{X}; M) = \begin{cases} 
\frac{x \cdot J(M)^{-1} x}{\text{det}(J(M))} & x = \text{supp}(J(M)), \\
+\infty & \text{otherwise}, 
\end{cases}
\]
(15)
The matrix \( J(M) \) varies with varying the POVM, but it is bounded from above by the quantum Cramér-Rao inequality [25]:
\[
J(M) \leq J_Q,
\]
(16)
\[
\Leftrightarrow J(M)^{-1} \geq J_Q^{-1},
\]
(17)
where \( J_Q \) is the quantum Fisher information matrix [26], which is a monotone metric on the quantum state space with the coordinate system \( \theta \). The quantum Fisher information matrix is not uniquely determined, but from the monotonicity there exist the minimum \( J_L \) and the maximum \( J_R \) [27], where \( J_L \) (\( J_R \)) is the symmetric (right) logarithmic derivative Fisher information matrix. Their \( ij \) elements are defined as
\[
[J_L]_{ij} := \frac{i}{2} \{ \{ \hat{L}_i, \hat{L}_j \} \},
\]
(18)
\[
[J_R]_{ij} := \{ \{ \hat{L}_i, \hat{L}_j \} \},
\]
(19)
where the curly brackets \( \{ \} \) denote the anticommutator, and a Hermitian \( \hat{L}_i \) and a non-Hermitian \( \hat{L}_i \) are defined to be the solution to
\[
\partial_i \hat{\rho} = \frac{i}{2} \{ \hat{\rho}, \hat{L}_i \}.
\]
(20)
It can be shown (see Appendix A) that
\[
[J_L]_{ij} = C_i(\hat{\lambda}_i, \hat{\lambda}_j),
\]
(22)
\[
[J_R]_{ij} = C(\hat{\lambda}_i, \hat{\lambda}_j),
\]
(23)
where
\[
C_i(\hat{\lambda}, \hat{\lambda}) := \frac{i}{2} \{ (\hat{\lambda}, \hat{\lambda}) \} - \{ \hat{\lambda}, \hat{\lambda} \},
\]
(24)
\[
C(\hat{\lambda}, \hat{\lambda}) := \{ \{ \hat{\lambda}, \hat{\lambda} \} \}
\]
(25)
are the symmetrized and unsymmetrized correlation functions of two observables. From the classical and quantum Cramér-Rao inequalities and
\[
x \cdot J_S^{-1} x = x \cdot J_Q^{-1} x = (\Delta X)^2,
\]
(26)
we find that the inequality
\[
\epsilon(\hat{X}; M) = \frac{x \cdot J(M)^{-1} x}{\text{det}(J(M))} \geq 0
\]
(27)
is satisfied for any quantum Fisher information.

We note that if the density matrix does not have full rank, for example, if the state is pure, the Cramér-Rao inequality (13) does not hold and needs a correction for the unbiasedness of the estimator due to the semi-positivity of the quantum states [28]. The measurement error defined in (7) satisfies (15) only for those states whose density matrices have full rank. However, all non-full-rank states can be approximated by full-rank states with arbitrary precision, and all states that can be generated realistically are mixed. Therefore, in the following, we consider only the case in which the density matrix has full rank.

III. TRADE-OFF RELATIONS ON MEASUREMENT
ERROR FOR AN ARBITRARY MEASUREMENT

The first result in this paper is the following theorem:

**Theorem 1.** For all observables \( \hat{X}_1, \hat{X}_2 \) and quantum states \( \hat{\rho} \), an arbitrary POVM \( M \) satisfies
\[
\epsilon(\hat{X}_1; M) \epsilon(\hat{X}_2; M) \geq \frac{1}{4} \{ |\langle \hat{X}_1, \hat{X}_2 \rangle| \}^2.
\]
(28)
where the square brackets \([ \cdot ]\) denote the commutator.

**Proof.** From the quantum Cramér-Rao inequality [25], we have
\[
J(M)^{-1} - J_R^{-1} \geq 0.
\]
(29)
The matrix \( J(M) \) is real and symmetric, and \( J_R \) is Hermitian. Thus, the following inequality is satisfied for all observables \( \hat{X}_\mu = x_0, \mu I + x_\mu \cdot \hat{\lambda}, \) and \( k \in \mathbb{R}^d \):
\[
(x_1 + ikx_2)(J(M)^{-1} - J_R^{-1}x_1 + ikx_2) \geq 0
\]
\[
\Rightarrow k^2 \epsilon(\hat{X}_2; M) + k |\langle \hat{X}_1, \hat{X}_2 \rangle| + \epsilon(\hat{X}_1; M) \geq 0.
\]
(30)
It follows from this inequality that the discriminant of the quadratic polynomial on the LHS of (30) is always negative. Therefore, (28) is proved.

Heisenberg originally discussed the trade-off relation between the measurement error of an observable and the disturbance in another noncommuting observable caused by the measurement. From this argument, it can be expected that the trade-off relation between measurement errors exists. We
have proved this in the form of (28). Holevo proved a similar trade-off relation for position and momentum for the coherent state [8].

Equation (28) is satisfied for all quantum states and observables on any finite-dimensional Hilbert space. However, the equality in (28) cannot be achieved for all quantum states and observables [see the dash-dotted curve in Fig. 2(a)]. For example, for \( \hat{\rho} = I/d \),

\[
\langle [\hat{X}_1, \hat{X}_2] \rangle = 0
\]

for all observables \( \hat{X}_1 \) and \( \hat{X}_2 \), and thus the RHS of (28) vanishes. For a POVM \( \hat{M} \) to achieve the equality of (28), the POVM must satisfy

\[
\varepsilon(\hat{X}_1; \hat{M}) = 0, \quad \varepsilon(\hat{X}_2; \hat{M}) < +\infty.
\]

To satisfy (32), the POVM \( \hat{M} \) must be the projection measurement of \( \hat{X}_1 \). However, in this case, the error for \( \hat{X}_2 \) diverges and the conditions (33) cannot be satisfied unless \( \hat{X}_2 \) commutes with \( \hat{X}_1 \).

**IV. ATTAINABLE BOUND OF THE MEASUREMENT ERRORS**

A simple but not optimal way to estimate \( \langle \hat{X}_1 \rangle \) and \( \langle \hat{X}_2 \rangle \) is to perform one projection measurement \( \hat{P}_1 \) on \( n_1 \) samples and another projection measurement \( \hat{P}_2 \) on \( n_2 = n - n_1 \) samples. This measurement scheme is asymptotically equivalent to the POVM measurement that randomly performs those two projection measurements with probabilities \( q_1 = n_1/n \) and \( q_2 = 1 - n_1/n \). If \( \hat{X}_1 \) and \( \hat{X}_2 \) cannot be simultaneously block-diagonalizable, the measurement errors satisfy

\[
\varepsilon(\hat{X}_1; \hat{M}) \varepsilon(\hat{X}_2; \hat{M}) = (\Delta X_1^2)(\Delta X_2^2).
\]

However, this measurement scheme does not exploit possible correlations between the observables \( \hat{X}_1 \) and \( \hat{X}_2 \). To utilize them, it is sufficient to perform projection measurements of two observables \( \hat{Y}_v \) with probability \( q_v (v = 1, 2) \) with \( q_1 + q_2 = 1 \), where \( \hat{Y}_1 \) and \( \hat{Y}_2 \) are linear combinations of \( \hat{X}_1 \) and \( \hat{X}_2 \). Therefore, we consider the following classes of POVM.

First, we define a set of projection measurements \( \mathcal{P}_{\hat{X}_1, \hat{X}_2} \) as that of all projection measurements corresponding to the spectral decompositions of the observables that are linear combinations of \( \hat{X}_1 \) and \( \hat{X}_2 \):

\[
\mathcal{P}_{\hat{X}_1, \hat{X}_2} = \left\{ \hat{P} = \{\hat{P}_i\}_{i=1}^d \left| \varepsilon \alpha_i, \beta_i \in \mathbb{R}, \right. \alpha_i \hat{X}_1 + \beta_i \hat{X}_2 = \sum_i \beta_i \hat{P}_i \right\}.
\]

The set of the measurement schemes that probabilistically perform projection measurements with \( \mathcal{P}_{\hat{X}_1, \hat{X}_2} \) is defined as

\[
\mathcal{M}_{\text{random}} := \left\{ q_1 \hat{P}_1 + q_2 \hat{P}_2 \left| \hat{P}_1, \hat{P}_2 \in \mathcal{P}_{\hat{X}_1, \hat{X}_2}, \right. q_1, q_2 \geq 0, q_1 + q_2 = 1 \right\},
\]

where

\[
q_1 \hat{P}_1 + q_2 \hat{P}_2 = \{q_1 \hat{P}_{1,v}\}_{v=1}^d
\]

for \( \hat{P}_v = \{\hat{P}_{v,i}\}_{i=1}^d \).

In real experimental setups, measurements always suffer from noises which deteriorate the precision of projection measurement \( \hat{P} \). Such a noisy measurement can be expressed as

\[
\hat{M} = \hat{F} \hat{P} = \left\{ \sum_j F_{ij} \hat{P}_j \right\}_i,
\]

where \( \hat{F} \) is the so-called information processing matrix or probability transition matrix whose elements satisfy

\[
F_{ij} \geq 0, \quad \sum_i F_{ij} = 1.
\]

The measurements described by \( \hat{F} \hat{P} \) cover a broad class of experimentally realizable measurements. For example, a typical scheme of quantum nondemolition (QND) measurement belongs to this class [16–18]. We note that the noise of a measurement in this class is described by a classical noise that is characterized by a classical noisy channel with \( \hat{F}_{ij} \). We define a set of measurements,

\[
\mathcal{M}_{\text{noisy}} := \left\{ \hat{F} \hat{P} \left| \hat{M} \in \mathcal{M}_{\text{random}}, F_{ij} \geq 0, \sum_i F_{ij} = 1 \right. \right\},
\]

which include random measurements consisting of noisy projection measurements. Note that the classes of measurements described above satisfy

\[
\mathcal{M}_{\text{random}} \subset \mathcal{M}_{\text{noisy}} \subset \mathcal{M}_{\text{all}},
\]

where \( \mathcal{M}_{\text{all}} \) denotes the totality of POVM measurements.

The second main result in this paper is the following theorem:
Theorem 2. For all observables $\hat{X}_1$, $\hat{X}_2$ and quantum states $\hat{\rho}$, an arbitrary POVM $M \in M_{\text{noisy}}$ satisfies

$$\epsilon(\hat{X}_1; M) \epsilon(\hat{X}_2; M) \geq (\Delta_{Q}X_1)^2(\Delta_{Q}X_2)^2 - |C_Q(\hat{X}_1, \hat{X}_2)|^2.$$  \hfill (42)

Moreover, the measurements that achieve the equality of (42) exist for all quantum states and observables.

Here $\Delta_{Q}$ and $C_Q$ are defined as follows. Let $\mathcal{H}_A (a = 1, 2, \ldots)$ be the simultaneous irreducible invariant subspace of $\hat{X}_1$ and $\hat{X}_2$ ($\mathcal{H} = \oplus_a \mathcal{H}_a$), and $\hat{P}_a$ the projection operator on $\mathcal{H}_a$. We define the probability distribution as $p_a := \langle \hat{P}_a \rangle$ and the post-measurement state of the projection measurement $\{\hat{P}_1, \hat{P}_2, \ldots\}$ as

$$\hat{\rho}_a := \frac{1}{p_a} \hat{P}_a \hat{\rho} \hat{P}_a.$$  \hfill (43)

Then, $\Delta_{Q}$ and $C_Q$ are defined as

$$\Delta_{Q}(\hat{X}_1, \hat{X}_2) := \sum_a p_a \left(\left|\langle \hat{X}_1 \rangle_{a}^{\hat{\rho}_a} - \langle \hat{X}_2 \rangle_{a}^{\hat{\rho}_a}\right|^2\right)$$  \hfill (44)

$$C_Q(\hat{X}_1, \hat{X}_2) := \sum_a p_a \left(\langle \hat{X}_1 \rangle_{a}^{\hat{\rho}_a} \langle \hat{X}_2 \rangle_{a}^{\hat{\rho}_a} - \langle \hat{X}_1 \hat{X}_2 \rangle_{a}^{\hat{\rho}_a} \right).$$  \hfill (45)

where $\langle \hat{X}_a \rangle = \text{Tr}[\hat{X}_a \hat{\rho}]$. If $\hat{X}_1$ and $\hat{X}_2$ are simultaneously block-diagonalizable, then quantum fluctuations and correlations of observables are determined by the diagonal blocks of $\hat{\rho}$. (Note that $\langle \hat{X}_a \rangle$ is independent of the off-diagonal blocks of $\hat{\rho}$.) If two observables commute with each other, the RHS of (45) vanishes.

For qubits (dim $\mathcal{H} = 2$), (42) can be proven for all POVM measurements, as stated in the following theorem:

Theorem 3. For all quantum states $\hat{\rho}$ and observables $\hat{X}_a$ on the two-dimensional Hilbert space, (42) is satisfied for all POVMs $M \in M_{\text{noisy}}$.

Inequality (42) is stronger than (28) and the trade-off relations obtained by Nagaoka [12] [see Fig. 2(a)], and it reduces to the trade-off relation found in Ref. [13] for dim $\mathcal{H} = 2$ and $\hat{\rho} = \hat{I}/2$. The optimal measurement of Englert’s complementarity [14] for dim $\mathcal{H} = 2$ achieves the bound by (42).

We emphasize that the bound set by (42) can be achieved for all quantum states and observables, whereas the bound set by (28) cannot. For example, for

$$\hat{\rho} = \frac{r}{2S+1} \hat{I} + (1-r) |S\rangle \langle S|,$$

$$\hat{X}_1 = \hat{S}_z, \quad \hat{X}_2 = \hat{S}_x \cos \varphi + \hat{S}_y \sin \varphi,$$

$$q_1 = q_2 = 1/2,$$

the measured observable

$$\hat{y}_v = \hat{S}_x \cos y_v + \hat{S}_y \sin y_v$$

is determined by the solution to

$$\cos \varphi + \cos \varphi \cos^2(y_1 - y_2) - 2 \cos(y_1 + y_2 - \varphi) \times \cos(y_1 - y_2) = 0,$$

where $\hat{S}_i$ is the spin operator of total spin $S$ in the $i (= x, y, z)$ direction, and $|S\rangle$ is the eigenstate of $\hat{S}_z$ with eigenvalue $S$. The RHS of (28) and that of (42) are given by $|\langle \hat{S}_1 | S \rangle|^2$ and $|rS(2S-1)/6 + S/2|^2 \sin^2 \varphi$, respectively. Such an optimal measurement can be implemented, for example, by using cold atoms [16–19]. By letting an ensemble of atoms interact with a linearly polarized off-resonant laser whose propagation direction is parallel to the direction specified by $y_v$ in $\hat{I}_v$ [see Fig. 2(b)], the angle of a paramagnetic Faraday rotation of the laser polarization carries information about $\langle \hat{y}_v \rangle$. The rotation angle can be detected by a polarimeter using a polarization-dependent beam splitter. If the intensity of the laser is sufficiently strong, this scheme achieves the projection measurement of $\hat{y}_v$.

In the following, we prove Theorems 2 and 3:

Proof of Theorem 2. If two POVMs satisfy $M' = FM$ with an information processing matrix $F$, they satisfy

$$J(M') \leq J(M).$$  \hfill (51)

Hence, we only have to consider the case in which $M \in M_{\text{random}}$.

Let $\hat{X}_a = x_{a,0} \hat{I} + x_{a,1} \hat{\lambda}$ be a linear combination of $\hat{Y}_v = y_{v,0} \hat{I} + y_{v,1} \hat{\lambda}$ ($v = 1, 2$), and $A = (a_{\mu\nu})$ be its coefficient:

$$\hat{X}_a = \sum \nu a_{\mu\nu} \hat{Y}_\nu.$$  \hfill (52)

We consider the POVM measurement $M = q_1 P_1 + q_2 P_2 \in M_{\text{random}}$, where $P_\nu = \{\hat{P}_\nu\}$ corresponds to the spectral decompositions of the observables $\hat{Y}_\nu = \sum \nu \beta_{\nu}, \hat{P}_\nu$.

As shown in Appendix A, the inverse of $J(M)$ can be obtained as

$$y_{v} \cdot J(M)^{-1} y_{v} = (\Delta_{Q}Y_{v})^2 + (q_{v} - 1)(\Delta_{Q}Y_{v})^2,$$

$$y_{1} \cdot J(M)^{-1} y_{2} = C_s(\hat{Y}_1, \hat{Y}_2) - C_Q(\hat{Y}_1, \hat{Y}_2).$$

Let

$$J(M) := [R^T J(M)^{-1} R]^{-1},$$

$$\tilde{J}_s := [R^T J_s^{-1} R]^{-1},$$

$$\tilde{e}(M) := J(M)^{-1} - \tilde{J}_s^{-1}$$

be $2 \times 2$ matrices, where

$$R := (x_1 \ x_2) \quad (58)$$

is a $(d^2 - 1) \times 2$ matrix. From (52), (53), and (54), we obtain

$$\tilde{J}_s = \left(\frac{(\Delta_{Q}X_1)^2}{C_s(\hat{X}_1, \hat{X}_2)} \ C_s(\hat{X}_1, \hat{X}_2) \right)$$

$$A = \left(\frac{(\Delta_{Q}Y_1)^2}{C_Q(\hat{Y}_1, \hat{Y}_2)} \ C_Q(\hat{Y}_1, \hat{Y}_2) \right) A^T,$$

$$\tilde{e}(M) = A \left(\frac{q_{v}}{q_{v}}(\Delta_{Q}Y_1)^2 - C_Q(\hat{Y}_1, \hat{Y}_2) \frac{q_{v}}{q_{v}}(\Delta_{Q}Y_2)^2\right) A^T.$$  \hfill (60)

The measurement error of the observable $\hat{X}_a$ can be written as

$$e(\hat{X}_a; M) = [\tilde{e}(M)]_{aa}.$$  \hfill (61)

Because $\tilde{e}(M)$ is symmetric, we have

$$e(\hat{X}_1; M) \geq \text{det}[\tilde{e}(M)]$$

$$= \text{det} \left(\frac{(\Delta_{Q}Y_1)^2}{C_Q(\hat{Y}_1, \hat{Y}_2)} \ (\Delta_{Q}Y_2)^2\right) (\text{det} A)^2$$

$$= (\Delta_{Q}X_1)^2(\Delta_{Q}X_2)^2 - C_Q(\hat{X}_1, \hat{X}_2)^2.$$
The condition for the equality to hold is that the off-diagonal elements of $\tilde{\epsilon}(M)$ vanish. The observables $\hat{Y}_\epsilon$ that satisfy this condition exist for all $\epsilon$.  

Next we prove (42) for the two-dimensional Hilbert space. We first prove the following two lemmas: 

**Lemma 1** For all POVM $M \in \mathcal{M}_{\text{all}}$, 

$$\text{Tr}[J(M)J^{-1}_s] \leq d - 1$$  

(63) 

is satisfied. This lemma was also shown in Ref. [29]. 

**Proof.** Let the spectral decomposition of each element of $M = \{M_1\}$ be 

$$\hat{M}_i = \sum_j k_{ij} |\psi_{ij}\rangle \langle \psi_{ij}|,$$  

(64) 

and we define an associated POVM, 

$$N = \{\hat{N}_{ij} = k_{ij} |\psi_{ij}\rangle \langle \psi_{ij}|\}_{i,j}.$$  

(65) 

From the fact that there always exists an information processing matrix $F$ such that $M = FN$, 

$$J(M) \leq J(N)$$  

(66) 

is satisfied. By denoting 

$$\hat{N}_{ij} = k_{ij} \hat{i} + v_{ij} \cdot \hat{k}$$  

(67) 

from the facts that 

$$\hat{N}_{ij} = k_{ij} \hat{N}_{ij}, \quad \sum_j k_{ij} = d,$$  

(68) 

we obtain 

$$\text{Tr}[J(M)J^{-1}_s] \leq \text{Tr}[J(N)J^{-1}_s] = \sum_{ij} p_{ij}^{-1} v_{ij} \cdot J^{-1}_s v_{ij}$$  

$$= \sum_{ij} (\Delta \hat{N}_{ij})^2 = \sum_{ij} k_{ij} (\hat{N}_{ij} - \langle \hat{N}_{ij} \rangle)^2 \langle \hat{N}_{ij} \rangle$$  

$$= \sum_{ij} (k_{ij} - \langle \hat{N}_{ij} \rangle) = d - 1.$$  

(69) 

Lemma 2. Let $K := J^{-1/2}_s J(M) J^{-1/2}_s$ For the two-dimensional Hilbert space ($d = 2$), the following inequalities hold: 

$$\text{Tr}[K] \leq 1 \iff \det[K^{-1} - I] \geq 1.$$  

(70) 

**Proof.** Because 

$$P := J(M)^{-1/2} R [R^T J(M)^{-1} R]^{-1} R^T J(M)^{-1/2}$$  

(71) 

is a projection matrix ($P^2 = P \leq I$), we have 

$$\text{Tr}[K] = \text{Tr}[R^T J(M)^{-1} R]^{-1} R^T J(M)^{-1/2} = \text{Tr}[P J(M)^{1/2} J^{-1}_s J(M)^{1/2}]$$  

$$\leq \text{Tr}[J(M)J^{-1}_s].$$  

(72) 

Therefore, from Lemma 1, the statement is proved. 

**Proof of Theorem 3.** If $\hat{X}_1$ and $\hat{X}_2$ commute with each other, 

$$(\Delta Q X_p)^2 = C_Q(\hat{X}_1, \hat{X}_2) = 0,$$  

(73) 

and therefore the RHS of (42) vanishes. Hence, we have only to consider the case in which $\hat{X}_1$ and $\hat{X}_2$ do not commute. It follows from the fact that $\tilde{\epsilon}(M)$ is symmetric and from Lemma 2 that 

$$\epsilon(\hat{X}_1; M) \epsilon(\hat{X}_2; M) \geq \det[\tilde{\epsilon}(M)]$$  

$$= \det[K^{-1} - I] \det[J^{-1}_s] \geq \det[J^{-1}_s]$$  

$$= (\Delta X_1)^2 (\Delta X_2)^2 - C_Q(\hat{X}_1, \hat{X}_2)^2.$$  

(74) 

From the facts that 

$$\Delta Q X_p = \Delta X_p,$$  

(75) 

(42) is proved. 

**V. NUMERICAL RESULTS OF FINDING ACHIEVABLE BOUND FOR ALL POVM MEASUREMENTS** 

Our trade-off relation (42) is rigorously proven for the measurements in $\mathcal{M}_{\text{all}}$ for $\dim H = 2$ and $\mathcal{M}_{\text{sym}}$ for $\dim H \geq 3$. For higher-dimensional Hilbert spaces from $\dim H = 3$ to 7, we numerically calculate the measurement errors of $10^3$ randomly chosen POVMs in $\mathcal{M}_{\text{all}}$ for randomly chosen 10 pairs of quantum states and 2 observables ($\hat{\rho}, \hat{X}_1, \hat{X}_2$). We find that the calculated measurement errors satisfy (42); A typical example of the numerical calculation is shown in Fig. 2 (a). The area within the bound is blacked out by $10^3$ data points with no point found outside of the bound. The range $\dim H = 3$ to 7 includes prime numbers ($\dim H = 3, 5, 7$), a power of prime ($\dim H = 4$), and a composite number that is not a power of prime ($\dim H = 6$). Therefore, we conjecture the following: 

**Conjecture 1.** For all observables $\hat{X}_1, \hat{X}_2$ and quantum states $\hat{\rho}$, all POVMs $M \in \mathcal{M}_{\text{all}}$ satisfy (42). 

**VI. CONCLUSION AND DISCUSSION** 

To summarize, we have formulated the complementarity of quantum measurement in a finite-dimensional Hilbert space by invoking quantum estimation theory. To quantify the information retrieved by the measurement, it is essential to take into account the estimation process. We have shown that the measurement errors of noncommuting observables satisfy the Heisenberg-type uncertainty relation, and find the stronger bound (42) that can be achieved for all quantum states and observables. The measurement schemes that achieve this bound can be implemented experimentally in cold-atom systems. 

The bound set by (42) is proved for the measurement schemes that perform two projection measurements probabilistically. We numerically show that randomly generated POVM measurements satisfy the bound. Thus, we conjecture that (42) is satisfied for all quantum measurements. The rigorous proof of the conjecture remains a future problem. 

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APPENDIX: INVERSE OF FISHER INFORMATION MATRIX

In this section, we show how to calculate the inverses of classical and quantum Fisher information matrices. First, we show $J(M)^{-1}$. By expanding each element of $M$ as
\[ \hat{M}_k = r_k + v_k \cdot \hat{\lambda}, \]  
the Fisher information matrix can be expressed as
\[ J(M) = \sum_{k=1}^{m} \frac{v_k v_k^T}{p_k} = VP^{-1}V^T, \]  

where $V$ and $P$ defined by
\[ V = (v_1 \cdots v_m), \]  
\[ P = \text{diag}(p_1, \ldots, p_m) \]
are $(d^2 - 1) \times m$ and $m \times m$ real matrices, respectively. The inverse $J(M)^{-1}$ can be obtained as
\[ J(M)^{-1} = (V^T)^{-1} [P - PA(A^{-1}PA)^{-1}A^{-1}P] V^{-1}, \]  
where $A$ is a real matrix whose column vectors $a_i$ are linearly independent and satisfy $VA = 0$. Then, $A$ satisfies
\[ VA = VA^T = 0, \]  
\[ V^{-1}V = V^T(V^T)^{-1} = I - AA^{-1}, \]  
\[ A^{-1} = A^T(A^T)^{-1} = I. \]

Next, we show (22) and (23). By expanding $\hat{L}_i$ as
\[ \hat{L}_i = a_i \hat{I} + b_i \cdot \hat{\lambda}, \]
and from (20), we obtain
\[ a_i + \theta \cdot b_i = 0, \]  
\[ a_i \theta + G_{\theta} b_i = e_i, \]  
where $e_i$ is a unit vector whose $i$th element is 1, and $G_{\theta}$ is a symmetric matrix whose $ij$ element is defined as
\[ [G_{\theta}]_{ij} := \frac{1}{2} \{ (\hat{\lambda}_i, \hat{\lambda}_j) \}. \]

Therefore, the symmetric logarithmic derivative Fisher information can be written as
\[ J_S = (G_{\theta} - \theta \theta^T)^{-1}, \]  
and its inverse can be obtained as (22).

To derive (23), we expand $\hat{L}_i$ as
\[ \hat{L}_i = c_i \hat{I} + d_i \cdot \hat{\lambda}. \]

Since $\hat{L}_i$ is non-Hermitian, $c_i$ is complex and $d_i$ is a complex vector. From (21), these coefficients satisfy
\[ c_i + \theta \cdot d_i = 0, \]  
\[ c_i \theta + (G_{\theta} - F_{\theta}) d_i = e_i, \]
where $F_{\theta}$ is a Hermitian matrix whose $ij$ element is defined as
\[ [F_{\theta}]_{ij} := \frac{1}{2} \{ (\hat{\lambda}_i, \hat{\lambda}_j) \}. \]

Therefore, the right logarithmic derivative Fisher information can be written as
\[ J_R = (G_{\theta} + F_{\theta} - \theta \theta^T)^{-1}, \]  
and (23) is derived.